1. INTRODUCTION

OVERVIEW

These tables give information about all finite groups whose order is at most 100, with the exception of the groups of orders 64 and 96. For each group a presentation is given and, as well, where possible, a direct product decomposition. The sizes of the conjugacy classes and order of their elements are given. For groups whose orders are less than 72 the elements of the conjugacy classes are listed in terms of the generators.

For groups whose order is less than 72, the character table is shown, together with the kernels of the irreducible representations. A list of normal subgroups, as a union of conjugacy classes is then provided, together with an expression as an intersection of these irreducible kernels. The normal subgroups are identified up to isomorphism, as are the corresponding quotient groups. For groups of order 72 or more the character table is not given but the normal subgroups, and their quotients, are listed up to isomorphism. Special normal subgroups are identified, such as the terms of the ascending central series, the derived series, the norm and the Frattini subgroup. Where appropriate the Sylow subgroups are given, as well as the maximal subgroups.

COMPUTATIONAL SOFTWARE

There are several computational packages for finite groups. **GAP** (Groups, Algorithms and Programming) was created by Joachim Neubuser from 1986 and is now maintained by several GAP centres, such as the one at the University of St Andrews. **CAYLEY** was created by John Cannon for group theory, but was later expanded to **MAGMA** to include other computational abstract algebra. It is maintained by the Computational Algebra Group at the University of Sydney.

This atlas complements them by providing an overview, in the same way that a printed map can serve different purposes to a GPS. I have based it ON *Die Untergruppenverbande der Gruppen der Ordnung ≤ 100 mit Ausnahme der Ordnung 64 und 96*, a postdoctoral thesis by Joachim Neubüser, Kiel (1967). This work did not include character tables and here I have computed them by hand. They are independent of the above software packages, and it would be useful to check my character tables using them. No doubt there are some errors in my tables, though in the early days of CAYLEY I was able to point out at least one error in one of CAYLEY's results. I imagine that, while my atlas contains an occasional error, MAGMA is by now 100% reliable.

GROUP NUMBERS

The groups are numbered in the form N.*m* meaning the m^{th} group of order N, except that N.1, the cyclic group of order N, is numbered simply as N. The first group of each order is the cyclic group, followed by the other abelian groups. The non-abelian groups that follow are in the same order as in Neubüser's thesis. With a few exceptions this is the same numbering as in *Group Tables* by A.D. Thomas and G.V.Wood, Shive Publishing, Kent UK (1980) which gives information on groups up to order 32.

NAMES

Many of the groups are also given symbolic names. The cyclic group of order N is written as just N and groups that are direct products of smaller groups are indicated. The notation G^n denotes the direct product of *n* copies of G, so that 4×2^3 is the direct product of the cyclic group of order 4 and three copies of the cyclic group of order 2.

Other names are as follows.

$\mathbf{D}_{2n}$ is the dihedral group of order $2n = \langle A^n, B^2, BA = A^{-1}B \rangle$.

If n is odd, $\mathbf{D}_{4n} = \mathbf{D}_{2n} \times 2$.

$\mathbf{D}_{n,2m} = \langle A^n, B^{2m}, BA = A^{-1}B \rangle$ so $\mathbf{D}_{2n} = \mathbf{D}_{n,2}$.

$\mathbf{Q}_{4n}$ is the dicyclic group $\langle A^n, B^2 = A^n, BA = A^{-1}B \rangle$. ($\mathbf{Q}_8$ is the quaternion group.)

$\mathbf{M}_{m,n}^{(r)}$ = is the metacyclic group $\langle A, B \mid A^m = B^n = 1, BA = A^rB \rangle$ where $r^n \equiv 1 (\mathrm{mod}\ m)$.

$\mathbf{S}_n$ = the symmetric group of degree n.

$\mathbf{A}_n$ = the alternating group of degree n.

In the following cases q is a prime power and the elements are invertible $n \times n$ matrices over the field $\mathbf{GF}(q)$. If q is prime, $\mathbf{GF}(q) = \mathbb{Z}_q$.

$\mathbf{GL}(n, q)$ is the general linear group over $GF(q)$.

$\mathbf{SL}(n, q)$ is the special linear group, over $GF(q)$ (matrices with determinant 1).

$\mathbf{PSL}(n, q)$ is $\mathbf{SL}(n,q)/Z$ where $Z = \{\text{scalar matrices in } \mathbf{SL}(n,q)\}$.

PRESENTATIONS

I don't list the generators. They are all of the variables that appear in the relations. Relations of the form R = 1 are given as just R. I also omit commuting relations. So $\langle A^4, B^2{=}A^2, C^2, BA{=}A^{-1}B \rangle$ represents the presentation

$$\langle A, B, C \mid A^4 = 1, B^2 = A^2, C^2 = 1, BA = A^{-1}B, CA = AC, CB = BC \rangle$$

For every soluble group (in these tables, all except A_5 are soluble) the presentation has the form:

$$\langle A_1, A_2, ..., A_r \mid A_i^{n_i} = P_i,\ A_jA_i = C_{ij}A_iA_j \text{ (for } i, j = 1, ..., r \text{ and } i < j) \rangle$$

where each P_i is a word in $\{A_k$ for $k < i\}$ and each C_{ij} is a word in

$$\{A_k \text{ for } k < i < j\}.$$

Every element is expressible uniquely as a word $A_1^{a_1}A_2^{a_2} \ldots A_r^{a_r}$ where

$$0 \le a_i < n_i \text{ for each } i.$$

This presentation makes it easy to reduce any word into this form. The group order is $n_1n_2 \ldots n_r$.

THE TABLES

CHAPTER 2: ISOMORPHIC GROUPS

Many groups have more than one name. For example $\mathbf{D}_6 \cong \mathbf{S}_3$, $\mathbf{D}_{12} \cong \mathbf{D}_6 \times 2$ and $\mathbf{M}_{5,4}^{(3)} \cong \mathbf{M}_{5,4}^{(2)}$. This table lists the preferred name for each of these groups.

CHAPTER 3: IDENTIFYING GROUPS

The **order total** of a finite group is defined to be $\Psi = ½ \Sigma|g|$ where the sum is over all the non-identity elements. It is easy to show that this is always an integer. Although there are some cases of non-isomorphic groups, of the same order, having the same order total this is reasonably rare and so Ψ is a convenient way of identifying finite groups.

In this table groups of order up to 100 (except for prime cyclic groups and groups of orders 64 and 96) are listed first according to their order and then according to their order total. This is a convenient index to the other tables.

CHAPTER 4: ORDERS OF ELEMENTS AND CLASS EQUATIONS

For each group of order from 1 to 100 (except orders 64 and 96) I list the numbers of elements of each order (except order 1 of course) and the class equation in the form:

$$|G| = 1 + n_1{*}a_1 + n_2{*}a_2 + \ldots + n_k{*}a_k, \text{ where } 1 \le n_2 \le n_3 \le \ldots \le n_k,$$

meaning that there are a_r conjugacy classes of size n_r for each r. If a_r is omtted it is assumed to be 1.

C HAPTERS 5 – 17: CHARACTER TABLES, NORMAL SUBGROUPS etc

For convenience these are split into several chapters. Collectively they cover all groups of order up to 100, except for orders 64 and 96. For decomposable groups above order 16, only the presentation is given.

Where the elements of the conjugacy classes are listed an integer variable represents many powers of a generator. So, if A has order 8 and B has order 2, A^nC represents C, AC, A^2C, …, A^7C. The expression A^mB^nC represents C, AC, A^2C, …, A^7C, BC, ABC, A^2BC, …, A^7BC and $A^{2n+1}C$ represents AC, A^3C, A^5C, A^7C.

For non-abelian groups the conjugacy classes are normally given in terms of the generators. However, for groups that split as a direct product, the conjugacy classes are given in terms of pairs of conjugacy classes of the factors.

The main part of the character table gives the values of the irreducible characters on the conjugacy classes. The row above gives the sizes of the conjugacy classes (except for abelian groups) and the row below gives the orders of the elements in the conjugacy classes. The first column corresponds to the identity class and so gives the degrees of the irreducible representations. The characters are ordered by their degree. So χ_1 is always the trivial character. Then come the other linear characters, and so on.

To the right of the character table, in a column labelled 'ker' or just $\mathcal{K}$. I list the kernels of the respective irreducible characters. The numbers refer to the list of normal subgroups that follows. The kernel of χ_1 is, of course, G – the whole group. And where the kernel is trivial it is numbered 0, rather than 1, since 1 is the label for the first proper-non-trivial normal subgroup.

The symbol CC is an abbreviation for 'conjugacy cloass', but where space is limited this becomes 'C'. The symbol '#' denotes the size of the conjugacy class, '^' denotes the orders of the elements in a conjugacy class and $\mathcal{K}$ denotes the kernel of the representation corresponding to the irreducible character.

For each non-abelian group I list the proper, non-trivial normal subgroups in terms of the conjugacy classes so that 1 + 2 + 4, for example, will be the union of these conjugacy classes. These normal subgroups are numbered. The first few normal subgroups are the kernels of irreducible characters. Then come the intersections. In a column labelled '$\cap$' these intersections are shown. Since every normal subgroup is an intersection of kernels of irreducible representations, this will give all of the proper non-trivial normal subgroups. These are identified up to isomorphism, either by the group name, or by the alternative name. For example the group $\mathbf{C}_4 \times \mathbf{C}_2$ will be identified as 4×2, rather than as 8.2 and the dihedral group of order 8 appears as $\mathbf{D}_8$, rather than 8.4. The corresponding quotient group is also identified.

For non-abelian groups that are not *p*-groups the Sylow subgroups, and the maximal subgroups, are given up to isomorphism. The number of each, if greater than 1, is indicated by '$\times$ *n*'.

Certain special normal subgroups are identified in the normal subgroup table.

$\mathbf{G'}$ = the derived subgroup. If it is omitted for a non-abelian group it is assumed to be the whole group G. (Of course this only applies to $\mathbf{A}_5$ in these tables.) G'', G''' etc are shown if not 1.

$\mathbf{Z}$ = the centre. $\mathbf{Z}_2$, etc are the later terms in the ascending central series.

$\mathbf{\Phi}$ = the Frattini subgroup. the intersection of all the maximal subgroups.

$\mathfrak{N}$ = the norm, the intersection of the normalisers of the subgroups.